Delta-Nabla Optimal Control Problems*

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Abstract

We present a unified treatment to control problems on an arbitrary time scale by introducing the study of forward-backward optimal control problems. Necessary optimality conditions for delta-nabla isoperimetric problems are proved, and previous results in the literature obtained as particular cases. As an application of the results of the paper we give necessary and sufficient Pareto optimality conditions for delta-nabla bi-objective optimal control problems.

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1 INTRODUCTION

In order to deal with non-traditional applications in areas such as medicine, economics, or engineering, where the system dynamics are described on a time scale

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partly continuous and partly discrete, or to accommodate non-uniform sampled systems, one needs to work with systems defined on a so called time scale – see, e.g., [Atici et al. (2006)], [Atici and Uysal (2008)], [Malinowska and Torres (2010b)]. The optimal control theory on time scales was introduced in the beginning of the XXI century in the simpler framework of the calculus of variations, and is now a fertile area of research in control and engineering [Seiffertt et al. (2008)], [Malinowska and Torres (2010c)]. In the literature there are two different approaches to the problems of optimal control on time scales: some authors use the delta calculus [Bohner (2004)], [Bohner et al. (2010)], [Bartosiewicz and Torres (2008)], [Ferreira and Torres (2008)], [Malinowska et al. (2010)], [Malinowska and Torres (2009)], while others prefer the nabla methodology [Almeida and Torres (2009)], [Atici et al. (2006)], [Atici and Uysal (2008)], [Martins and Torres (2009)]. In this paper we propose a simple and effective unification of the delta and nabla approaches of optimal control on time scales. More precisely, we consider the problem of minimizing or maximizing a delta-nabla cost integral functional

$$\mathcal{L}(y) = \gamma_1 \int_a^b L_\Delta\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t + \gamma_2 \int_a^b L_\nabla\left(t, y^\rho(t), y^\nabla(t)\right) \nabla t \qquad (1)$$

subject to given boundary conditions and an isoperimetric constraint of the form

$$\mathcal{K}(y) = k_1 \int_a^b K_\Delta\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t + k_2 \int_a^b K_\nabla\left(t, y^\rho(t), y^\nabla(t)\right) \nabla t = k\,. \qquad (2)$$

Main results include Euler-Lagrange necessary optimality type conditions for delta-nabla isoperimetric problems (1)–(2) (see Section 3.1). Isoperimetric problems have found a broad class of important applications throughout the centuries. Concrete isoperimetric problems in engineering have been investigated by a number of authors – cf. [Almeida and Torres (2009b)], [Curtis (2004)], and references therein. Here, as an application of our results, we obtain the recent results of [Almeida and Torres (2009)], [Atici et al. (2006)], [Bohner (2004)], and [Ferreira and Torres (2010)] as straightforward corollaries. In Section 3.2 we consider delta-nabla bi-objective problems. Our more general approach to optimal control in terms of the delta-nabla problem (1)–(2) allows to obtain necessary and sufficient conditions for Pareto optimality. The results of the paper are illustrated by several examples.

2 PRELIMINARIES

We assume the reader to be familiar with the calculus on time scales. For an introduction to the subject we refer to the seminal papers [Aulbach and Hilger (1990)] and [Hilger (1990)], the nice survey [Agarwal et al. (2002)], and the books [Bohner and A. Peterson (2001)], [Bohner and A. Peterson (2003)], and [Lakshmikantham et al. (1996)].

Throughout the whole paper we assume $\mathbb{T}$ to be a given time scale with $a, b \in \mathbb{T}$, $a < b$, and we set $I := [a, b] \cap \mathbb{T}$ for $[a, b] \subset \mathbb{R}$. Moreover, we define $I_\kappa^\kappa := I^\kappa \cap I_\kappa$ with the standard notations $I^\kappa = I \setminus (\rho(b), b]$ and $I_\kappa = I \setminus [a, \sigma(a))$.

We recall some necessary results. If y is delta differentiable at $t \in \mathbb{T}$, then $y^\sigma(t) = y(t) + \mu(t) y^\Delta(t)$; if y is nabla differentiable at t, then $y^\rho(t) = y(t) - \nu(t) y^\nabla(t)$. If the functions $f, g : \mathbb{T} \to \mathbb{R}$ are delta and nabla differentiable with continuous derivatives, then the following formulas of integration by parts hold:

$$\begin{aligned}
\int_a^b f^\sigma(t) g^\Delta(t) \Delta t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\Delta(t) g(t) \Delta t \,, \\
\int_a^b f(t) g^\Delta(t) \Delta t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\Delta(t) g^\sigma(t) \Delta t \,, \\
\int_a^b f^\rho(t) g^\nabla(t) \nabla t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\nabla(t) g(t) \nabla t \,, \\
\int_a^b f(t) g^\nabla(t) \nabla t &= (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\nabla(t) g^\rho(t) \nabla t \,.
\end{aligned} \tag{3}$$

The following fundamental lemma of the calculus of variations on time scales, involving a nabla derivative and a nabla integral, was proved in [Martins and Torres (2009)].

Lemma 1. (The nabla Dubois-Reymond lemma – cf. Lemma 14 of [Martins and Torres (2009)]) *Let $f \in C_{ld}(I, \mathbb{R})$. If*

$$\int_a^b f(t) \eta^\nabla(t) \nabla t = 0 \quad \textit{for all} \quad \eta \in C_{ld}^1(I, \mathbb{R}) \quad \textit{such that} \quad \eta(a) = \eta(b) = 0 \,,$$

then $f(t) \equiv c$ for all $t \in I_\kappa$, where c is a constant.

Lemma 2 is the analogous delta version of Lemma 1.

Lemma 2. (The delta Dubois-Reymond lemma – cf. Lemma 4.1 of [Bohner (2004)]) *Let $g \in C_{rd}(I, \mathbb{R})$. If*

$$\int_a^b g(t) \eta^\Delta(t) \Delta t = 0 \quad \textit{for all } \eta \in C_{rd}^1(I, \mathbb{R}) \textit{ such that } \eta(a) = \eta(b) = 0,$$

then $g(t) \equiv c$ on I^κ for some $c \in \mathbb{R}$.

Proposition 3 gives a relationship between delta and nabla derivatives.

Proposition 3. (cf. Theorems 2.5 and 2.6 of [Atici and Guseinov (2002)]) *(i) If $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$ and f^Δ is continuous on $\mathbb{T}^\kappa$, then f is nabla differentiable on $\mathbb{T}_\kappa$ and*

$$f^\nabla(t) = \left(f^\Delta\right)^\rho(t) \quad \textit{for all } t \in \mathbb{T}_\kappa \,. \tag{4}$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}_\kappa$ and f^∇ is continuous on $\mathbb{T}_\kappa$, then f is delta differentiable on $\mathbb{T}^\kappa$ and

$$f^\Delta(t) = \left(f^\nabla\right)^\sigma(t) \quad \textit{for all } t \in \mathbb{T}^\kappa \,. \tag{5}$$

Proposition 4. (cf. Theorem 2.8 of [Atici and Guseinov (2002)]) *Let $a, b \in \mathbb{T}$ with $a \leq b$ and let f be a continuous function on $[a, b]$. Then,*

$$\int_a^b f(t)\Delta t = \int_a^{\rho(b)} f(t)\Delta t + (b - \rho(b))f^\rho(b)\,,$$
$$\int_a^b f(t)\Delta t = (\sigma(a) - a)f(a) + \int_{\sigma(a)}^b f(t)\Delta t\,,$$
$$\int_a^b f(t)\nabla t = \int_a^{\rho(b)} f(t)\nabla t + (b - \rho(b))f(b)\,,$$
$$\int_a^b f(t)\nabla t = (\sigma(a) - a)f^\sigma(a) + \int_{\sigma(a)}^b f(t)\nabla t\,.$$

We end our brief review of the calculus on time scales with a relationship between the delta and nabla integrals.

Proposition 5. (cf. Proposition 7 of [Gürses et al. (2005)]) *If function $f : \mathbb{T} \to \mathbb{R}$ is continuous, then for all $a, b \in \mathbb{T}$ with $a < b$ we have*

$$\int_a^b f(t)\Delta t = \int_a^b f^\rho(t)\nabla t\,, \tag{6}$$
$$\int_a^b f(t)\nabla t = \int_a^b f^\sigma(t)\Delta t\,. \tag{7}$$

3 MAIN RESULTS

Let $\mathbb{T}$ be a given time scale with $a, b \in \mathbb{T}$, $a < b$, and $\mathbb{T} \cap (a, b) \neq \emptyset$; $L_\Delta(\cdot,\cdot,\cdot)$ and $L_\nabla(\cdot,\cdot,\cdot)$ be two given smooth functions from $\mathbb{T} \times \mathbb{R}^2$ to $\mathbb{R}$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. Our results are trivially generalized for admissible functions $y : \mathbb{T} \to \mathbb{R}^n$ but for simplicity of presentation we restrict ourselves to the scalar case $n = 1$.

3.1 DELTA-NABLA ISOPERIMETRIC PROBLEMS

We consider the delta-nabla integral functional

$$\mathcal{L}(y) = \gamma_1 \int_a^b L_\Delta\left(t, y^\sigma(t), y^\Delta(t)\right)\Delta t + \gamma_2 \int_a^b L_\nabla\left(t, y^\rho(t), y^\nabla(t)\right)\nabla t\,.$$

For brevity we introduce the operators $[y]$ and $\{y\}$ defined by

$$[y](t) = \left(t, y^\sigma(t), y^\Delta(t)\right) \quad \text{and} \quad \{y\}(t) = \left(t, y^\rho(t), y^\nabla(t)\right)\,.$$

Then we can write:

$$\mathcal{L}_\Delta(y) = \int_a^b L_\Delta[y](t)\Delta t\,,$$

$$\mathcal{L}_\nabla(y) = \int_a^b L_\nabla\{y\}(t)\nabla t\,,$$

$$\mathcal{L}(y) = \gamma_1\mathcal{L}_\Delta(y) + \gamma_2\mathcal{L}_\nabla(y) = \gamma_1\int_a^b L_\Delta[y](t)\Delta t + \gamma_2\int_a^b L_\nabla\{y\}(t)\nabla t\,.$$

Let α, β, γ_1, γ_2, k, k_1, and k_2 be given real numbers. Let us denote by $C^1_\diamond(I,\mathbb{R})$ the class of functions $y : I \to \mathbb{R}$ with $(|\gamma_1| + |k_1|)y^\Delta$ continuous on I^κ and $(|\gamma_2| + |k_2|)y^\nabla$ continuous on I_κ. We consider the question of finding $y \in C^1_\diamond(I,\mathbb{R})$ that is a solution to the problem

$$\text{extremize} \quad \mathcal{L}(y) = \gamma_1\int_a^b L_\Delta[y](t)\Delta t + \gamma_2\int_a^b L_\nabla\{y\}(t)\nabla t \tag{8}$$

subject to the boundary conditions

$$y(a) = \alpha\,, \quad y(b) = \beta\,, \tag{9}$$

and the isoperimetric constraint

$$\mathcal{K}(y) = k_1\int_a^b K_\Delta[y](t)\Delta t + k_2\int_a^b K_\nabla\{y\}(t)\nabla t = k\,, \tag{10}$$

where $K_\Delta(\cdot,\cdot,\cdot)$ and $K_\nabla(\cdot,\cdot,\cdot)$ are given smooth functions from $\mathbb{T}\times\mathbb{R}^2$ to $\mathbb{R}$.

Function $y \in C^1_\diamond(I,\mathbb{R})$ is said to be *admissible* provided it satisfies conditions (9) and (10). We are interested to obtain necessary conditions for an admissible function to be a local minimizer (or a local maximizer) to problem (8)–(10).

Definition 6. *We say that $\hat{y} \in C^1_\diamond(I,\mathbb{R})$ is a local minimizer (respectively local maximizer) to problem* (8)–(10) *if there exists $\delta > 0$ such that $\mathcal{L}(\hat{y}) \le \mathcal{L}(y)$ (respectively $\mathcal{L}(\hat{y}) \ge \mathcal{L}(y)$) for all admissible functions $y \in C^1_\diamond(I,\mathbb{R})$ satisfying the inequality $\| y - \hat{y} \|_{1,\infty} < \delta$, where $\| y \|_{1,\infty} := \| y^\sigma \|_\infty + \| y^\rho \|_\infty + \| y^\Delta \|_\infty + \| y^\nabla \|_\infty$ with $\| y \|_\infty := \sup_{t\in I^\kappa_\kappa} | y(t) |$.*

Let $\partial_i K$ denote the standard partial derivative of a function $K(\cdot,\cdot,\cdot)$ with respect to its ith variable, $i = 1, 2, 3$. The following definition is motivated by the time scale Euler-Lagrange equations proved in [Girejko et al. (2010)] and [Malinowska and Torres (2010)].

Definition 7. *We say that $\hat{y} \in C^1_\diamond(I,\mathbb{R})$ is an* extremal *of*

$$\mathcal{K}(y) = k_1\int_a^b K_\Delta[y](t)\Delta t + k_2\int_a^b K_\nabla\{y\}(t)\nabla t$$

if $\hat{y}$ satisfies the following Euler-Lagrange delta-nabla integral equations:

$$k_1\left(\partial_3 K_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right) + k_2\left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^{t} \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t \in I_\kappa\,;$$

$$k_1\left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^{t} \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right) + k_2\left(\partial_3 K_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \quad \forall t \in I^\kappa\,.$$

An extremizer (i.e., a local minimizer or a local maximizer) to problem (8)–(10) *that is not an extremal of $\mathcal{K}$ in* (10) *is said to be a normal extremizer; otherwise (i.e., if it is an extremal of $\mathcal{K}$), the extremizer is said to be abnormal.*

Remark 8. *The word* extremal *means "solution of the Euler-Lagrange necessary optimality conditions". An extremizer is an extremal; but an extremal is not necessarily an extremizer (it is just a candidate to extremizer given by the first order necessary conditions).*

Associated to problem (8)–(10) we introduce the following notations:

$$\begin{aligned} &H_\Delta[\hat{y},\lambda](t) := H_\Delta(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t),\lambda) := \gamma_1 L_\Delta[\hat{y}](t) - k_1\lambda K_\Delta[\hat{y}](t)\\ &H_\nabla\{\hat{y},\lambda\}(t) := H_\nabla(t,\hat{y}^\rho(t),\hat{y}^\nabla(t),\lambda) := \gamma_2 L_\nabla\{\hat{y}\}(t) - k_2\lambda K_\nabla\{\hat{y}\}(t)\,. \end{aligned} \tag{11}$$

We look to H_Δ and H_∇ as functions of four independent variables, and we denote the partial derivatives of $H_\Delta(\cdot,\cdot,\cdot,\cdot)$ and $H_\nabla(\cdot,\cdot,\cdot,\cdot)$ with respect to their ith argument, $i = 1,2,3,4$, by $\partial_i H_\Delta$ and $\partial_i H_\nabla$ respectively.

Theorem 9 (Necessary optimality conditions for normal extremizers of a delta-nabla isoperimetric problem)**.** *If $\hat{y} \in C^1_\diamond(I,\mathbb{R})$ is a normal extremizer to the isoperimetric problem* (8)–(10)*, then there exists $\lambda \in \mathbb{R}$ such that $\hat{y}$ satisfies the following delta-nabla integral equations:*

$$\partial_3 H_\Delta[\hat{y},\lambda](\rho(t)) + \partial_3 H_\nabla\{\hat{y},\lambda\}(t) - \left(\int_a^{\rho(t)} \partial_2 H_\Delta[\hat{y},\lambda](\tau)\Delta\tau + \int_a^{t} \partial_2 H_\nabla\{\hat{y},\lambda\}(\tau)\nabla\tau\right) = const \quad \forall t \in I_\kappa\,; \tag{12}$$

$$\partial_3 H_\Delta[\hat{y},\lambda](t) + \partial_3 H_\nabla\{\hat{y},\lambda\}(\sigma(t)) - \left(\int_a^{t} \partial_2 H_\Delta[\hat{y},\lambda](\tau)\Delta\tau + \int_a^{\sigma(t)} \partial_2 H_\nabla\{\hat{y},\lambda\}(\tau)\nabla\tau\right) = const \quad \forall t \in I^\kappa\,, \tag{13}$$

where H_Δ and H_∇ are defined by (11)*.*

Proof. Consider a variation of $\hat{y}$, say $\bar{y} = \hat{y} + \varepsilon_1\eta_1 + \varepsilon_2\eta_2$, where $\eta_i \in C^1_\diamond(I,\mathbb{R})$ and $\eta_i(a) = \eta_i(b) = 0$, $i \in \{1,2\}$, and ε_i is a sufficiently small parameter (ε_1 and ε_2 must be such that $\| \bar{y} - \hat{y} \|_{1,\infty} < \delta$ for some $\delta > 0$). Here η_1 is an arbitrary fixed function and η_2 is a fixed function that will be chosen later. Define the real function

$$\bar{K}(\varepsilon_1,\varepsilon_2) = \mathcal{K}(\bar{y}) = k_1 \int_a^b K_\Delta[\bar{y}](t)\Delta t + k_2 \int_a^b K_\nabla\{\bar{y}\}(t)\nabla t - k.$$

We have

$$\begin{aligned}\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} &= k_1 \int_a^b \left(\partial_2 K_\Delta[\hat{y}](t)\eta_2^\sigma(t) + \partial_3 K_\Delta[\hat{y}](t)\eta_2^\Delta(t)\right)\Delta t \\ &\qquad + k_2 \int_a^b \left(\partial_2 K_\nabla\{\hat{y}\}(t)\eta_2^\rho(t) + \partial_3 K_\nabla\{\hat{y}\}(t)\eta_2^\nabla(t)\right)\nabla t\,.\end{aligned}$$

The first and third integration by parts formula in (3) give

$$\begin{aligned}&\int_a^b \partial_2 K_\Delta[\hat{y}](t)\eta_2^\sigma(t)\Delta t \\ &\quad = \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau \eta_2(t)|_{t=a}^{t=b} - \int_a^b \left(\int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_2^\Delta(t)\Delta t \\ &\quad = -\int_a^b \left(\int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_2^\Delta(t)\Delta t\end{aligned}$$

and

$$\begin{aligned}&\int_a^b \partial_2 K_\nabla\{\hat{y}\}(t)\eta_2^\rho(t)\nabla t \\ &\quad = \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau \eta_2(t)|_{t=a}^{t=b} - \int_a^b \left(\int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_2^\nabla(t)\nabla t \\ &\quad = -\int_a^b \left(\int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_2^\nabla(t)\nabla t\end{aligned}$$

since $\eta_2(a) = \eta_2(b) = 0$. Therefore,

$$\begin{aligned}\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} &= k_1 \int_a^b \left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_2^\Delta(t)\Delta t \\ &\qquad + k_2 \int_a^b \left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_2^\nabla(t)\nabla t. \qquad (14)\end{aligned}$$

Let

$$f(t) = \partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau$$

and

$$g(t) = \partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau.$$

We can then write equation (14) in the form

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} = k_1 \int_a^b f(t)\eta_2^\Delta(t)\Delta t + k_2 \int_a^b g(t)\eta_2^\nabla(t)\nabla t. \tag{15}$$

Transforming the delta integral in (15) to a nabla integral by means of (6), we obtain that

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} = k_1 \int_a^b f^\rho(t)(\eta_2^\Delta)^\rho(t)\nabla t + k_2 \int_a^b g(t)\eta_2^\nabla(t)\nabla t$$

and by (4)

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} = \int_a^b \left(k_1 f^\rho(t) + k_2 g(t)\right)\eta_2^\nabla(t)\nabla t.$$

As $\hat{y}$ is a normal extremizer, we conclude by Lemma 1 that there exists η_2 such that $\left.\frac{\partial \bar{K}}{\partial \varepsilon_2}\right|_{(0,0)} \neq 0$. Note that the same result can be obtained by transforming the nabla integral in (15) to a delta integral by means of (7), and then using Lemma 2. Since $\bar{K}(0,0) = 0$, by the implicit function theorem we conclude that there exists a function ε_2 defined in the neighborhood of zero such that $\bar{K}(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0$, i.e., we may choose a subset of variations $\bar{y}$ satisfying the isoperimetric constraint. Let us now consider the real function

$$\bar{L}(\varepsilon_1, \varepsilon_2) = \mathcal{L}(\bar{y}) = \gamma_1 \int_a^b L_\Delta[\bar{y}](t)\Delta t + \gamma_2 \int_a^b L_\nabla\{\bar{y}\}(t)\nabla t.$$

By hypothesis, $(0,0)$ is an extremal of $\bar{L}$ subject to the constraint $\bar{K} = 0$ and $\nabla\bar{K}(0,0) \neq \mathbf{0}$. By the Lagrange multiplier rule, there exists some real λ such that $\nabla(\bar{L}(0,0) - \lambda\bar{K}(0,0)) = \mathbf{0}$. Having in mind that $\eta_1(a) = \eta_1(b) = 0$, we can write

$$\begin{aligned}\left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)} &= \gamma_1 \int_a^b \left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_1^\Delta(t)\Delta t \\ &\quad + \gamma_2 \int_a^b \left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_1^\nabla(t)\nabla t \end{aligned} \tag{16}$$

and

$$\begin{aligned}\left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)} &= k_1 \int_a^b \left(\partial_3 K_\Delta[\hat{y}](t) - \int_a^t \partial_2 K_\Delta[\hat{y}](\tau)\Delta\tau\right)\eta_1^\Delta(t)\Delta t \\ &\quad + k_2 \int_a^b \left(\partial_3 K_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 K_\nabla\{\hat{y}\}(\tau)\nabla\tau\right)\eta_1^\nabla(t)\nabla t. \end{aligned} \tag{17}$$

Let

$$m(t) = \partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau$$

and

$$n(t) = \partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau.$$

Then equations (16) and (17) can be written in the form

$$\left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)} = \gamma_1 \int_a^b m(t)\eta_1^\Delta(t)\Delta t + \gamma_2 \int_a^b n(t)\eta_1^\nabla(t)\nabla t$$

and

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)} = k_1 \int_a^b f(t)\eta_1^\Delta(t)\Delta t + k_2 \int_a^b g(t)\eta_1^\nabla(t)\nabla t.$$

Transforming the delta integrals in the above equalities to nabla integrals by means of (6) and using (4), we obtain

$$\left.\frac{\partial \bar{L}}{\partial \varepsilon_1}\right|_{(0,0)} = \int_a^b \left(\gamma_1 m^\rho(t) + \gamma_2 n(t)\right)\eta_1^\nabla(t)\nabla t$$

and

$$\left.\frac{\partial \bar{K}}{\partial \varepsilon_1}\right|_{(0,0)} = \int_a^b \left(k_1 f^\rho(t) + k_2 g(t)\right)\eta_1^\nabla(t)\nabla t.$$

Therefore,

$$\int_a^b \eta_1^\nabla(t)\left\{\gamma_1 m^\rho(t) + \gamma_2 n(t) - \lambda\left(k_1 f^\rho(t) + k_2 g(t)\right)\right\}\nabla t = 0. \tag{18}$$

Since (18) holds for any η_1, by Lemma 1 we have

$$\gamma_1 m^\rho(t) + \gamma_2 n(t) - \lambda\left(k_1 f^\rho(t) + k_2 g(t)\right) = c$$

for some $c \in \mathbb{R}$ and all $t \in I_\kappa$. Hence, condition (12) holds. Equation (12) can also be obtained by transforming nabla integrals to delta integrals by means of (7) and then using Lemma 2. Equation (13) can be shown in a totally analogous way. □

Example 10. *(normal extremals) (a) Let $\mathbb{T} = \{1, 3, 4\}$ and consider the problem*

$$\text{minimize} \;\; \mathcal{L}(y) = \int_1^4 t\left(y^\nabla(t)\right)^2 \nabla t \tag{19}$$

$$y(1) = 0, \quad y(4) = 1 \tag{20}$$

subject to the constraint

$$\mathcal{K}(y) = \int_1^4 t\left(y^\Delta(t)\right)^2 \Delta t = \frac{105}{242}. \tag{21}$$

Since $L_\nabla = t\left(y^\nabla\right)^2$ *and* $K_\Delta = t\left(y^\Delta\right)^2$, *we have*

$$\partial_2 L_\nabla = 0, \quad \partial_3 L_\nabla = 2ty^\nabla, \quad \partial_2 K_\Delta = 0, \quad \partial_3 K_\Delta = 2ty^\Delta.$$

Let us assume for the moment that we are in conditions to apply Theorem 9. Applying equation (13) *of Theorem 9 we get the following delta-nabla differential equation:*

$$2\sigma(t)y^\nabla(\sigma(t)) - \lambda 2ty^\Delta(t) = C, \quad t \in \{1,3\},$$

where $C \in \mathbb{R}$. *By* (5) *we can write the above equation in the form*

$$2\sigma(t)y^\Delta(t) - \lambda 2ty^\Delta(t) = C, \quad t \in \{1,3\}. \tag{22}$$

Since $y^\Delta(1) = (y(3) - y(1))/2 = y(3)/2$ *and* $y^\Delta(3) = y(4) - y(3) = 1 - y(3)$, *solving equation* (22) *subject to the boundary conditions* $y(1) = 0$ *and* $y(4) = 1$ *we get*

$$\begin{cases} 3y(3) - \lambda y(3) = C \\ 8(1 - y(3)) - 6\lambda(1 - y(3)) = C, \end{cases}$$

what implies

$$y(t) = \begin{cases} 0 & \text{if } t = 1 \\ \frac{8-6\lambda}{11-7\lambda} & \text{if } t = 3 \\ 1 & \text{if } t = 4. \end{cases} \tag{23}$$

Substituting (23) *into* (21) *we obtain* $\lambda_1 = \frac{-11}{3}$, $\lambda_2 = \frac{143}{21}$. *Hence, we get two extremals,* y_1 *and* y_2, *corresponding to* λ_1 *and* λ_2, *respectively:*

$$y_1(t) = \begin{cases} 0 & \text{if } t = 1 \\ \frac{9}{11} & \text{if } t = 3 \\ 1 & \text{if } t = 4 \end{cases}, \quad y_2(t) = \begin{cases} 0 & \text{if } t = 1 \\ \frac{69}{77} & \text{if } t = 3. \\ 1 & \text{if } t = 4 \end{cases}$$

One can easily check that $\mathcal{L}(y_1) = \frac{25}{22}$ *and* $\mathcal{L}(y_2) = \frac{1345}{1078}$. *We now show that* y_1 *is not an extremal for* $\mathcal{K}$. *Indeed,*

$$\begin{aligned} \partial_3 K_\Delta[y_1](t) - \int_a^t \partial_2 K_\Delta[y_1](\tau)\Delta\tau + \partial_3 K_\nabla\{y_1\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 K_\nabla\{y_1\}(\tau)\nabla\tau \\ = \partial_3 K_\Delta[y_1](t) = 2ty_1^\Delta(t) = \begin{cases} \frac{9}{11} & \text{if } t = 1 \\ \frac{12}{11} & \text{if } t = 3. \end{cases} \end{aligned}$$

Thus y_1 *is a candidate local minimizer to problem* (19)–(21).
(b) Let $\mathbb{T} = \{1,3,4\}$ *and consider the problem*

$$\text{minimize} \quad \mathcal{L}(y) = \int_1^4 t\left(y^\Delta(t)\right)^2 \Delta t \tag{24}$$

$$y(1) = 0, \quad y(4) = 1 \tag{25}$$

subject to the constraint

$$\mathcal{K}(y) = \int_1^4 t\left(y^\nabla(t)\right)^2 \nabla t = \frac{25}{22}. \tag{26}$$

Proceeding analogously as before, we find

$$y_1(t) = \begin{cases} 0 & \text{if } t = 1 \\ \frac{9}{11} & \text{if } t = 3 \\ 1 & \text{if } t = 4 \end{cases}$$

as a candidate local minimizer to problem (24)–(26)*.*

As a particular case of Theorem 9 we obtain the following result:

Corollary 11 (Necessary optimality condition for normal extremizers of a delta isoperimetric problem – cf. Theorem 3.4 of Ferreira and Torres (2010))**.** *Suppose that the problem of minimizing*

$$J(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t$$

subject to the boundary conditions $y(a) = y_a$, $y(b) = y_b$, and the isoperimetric constraint

$$I(y) = \int_a^b g(t, y^\sigma(t), y^\Delta(t))\Delta t = l$$

has a local solution at $\hat{y}$ in the class of functions $y : [a, b] \to \mathbb{R}$ such that y^Δ exists and is continuous on $[a, b]^\kappa$, and that $\hat{y}$ is not an extremal for the functional I. Then, there exists a Lagrange multiplier constant λ such that $\hat{y}$ satisfies

$$\frac{\Delta}{\Delta t}\left[\partial_3 F(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))\right] - \partial_2 F(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) = 0 \text{ for all } t \in [a, b]^{\kappa^2}$$

with $F(t, x, v) = L(t, x, v) - \lambda g(t, x, v)$.

Proof. The result follows from Theorem 9 by considering the particular case $\gamma_1 = k_1 = 1$ and $\gamma_2 = k_2 = 0$. □

One can easily cover abnormal extremizers within our result by introducing an extra multiplier λ_0. Let

$$\begin{aligned} H_\Delta[\hat{y}, \lambda_0, \lambda](t) &:= H_\Delta(t, y^\sigma(t), y^\Delta(t), \lambda_0, \lambda) := \gamma_1\lambda_0 L_\Delta[\hat{y}](t) - k_1\lambda K_\Delta[\hat{y}](t) \\ H_\nabla\{\hat{y}, \lambda_0, \lambda\}(t) &:= H_\nabla(t, y^\rho(t), y^\nabla(t), \lambda_0, \lambda) := \gamma_2\lambda_0 L_\nabla\{\hat{y}\}(t) - k_2\lambda K_\nabla\{\hat{y}\}(t). \end{aligned} \tag{27}$$

Theorem 12 (Necessary optimality conditions for normal and abnormal extremizers of a delta-nabla isoperimetric problem)**.** *If $\hat{y} \in C^1_\diamond(I, \mathbb{R})$ is an extremizer to the isoperimetric problem* (8)–(10)*, then there exist two constants*

λ_0 and λ, not both zero, such that $\hat{y}$ satisfies the following delta-nabla integral equations:

$$\begin{aligned} &\partial_3 H_\Delta[\hat{y}, \lambda_0, \lambda](\rho(t)) + \partial_3 H_\nabla\{\hat{y}, \lambda_0, \lambda\}(t) \\ &- \int_a^{\rho(t)} \partial_2 H_\Delta[\hat{y}, \lambda_0, \lambda](\tau)\Delta\tau - \int_a^t \partial_2 H_\nabla\{\hat{y}, \lambda_0, \lambda\}(\tau)\nabla\tau = const \quad \forall t \in I_\kappa\,; \end{aligned} \tag{28}$$

$$\begin{aligned} &\partial_3 H_\Delta[\hat{y}, \lambda_0, \lambda](t) + \partial_3 H_\nabla\{\hat{y}, \lambda_0, \lambda\}(\sigma(t)) \\ &- \int_a^{t} \partial_2 H_\Delta[\hat{y}, \lambda_0, \lambda](\tau)\Delta\tau - \int_a^{\sigma(t)} \partial_2 H_\nabla\{\hat{y}, \lambda_0, \lambda\}(\tau)\nabla\tau = const \quad \forall t \in I^\kappa\,, \end{aligned} \tag{29}$$

where H_Δ and H_∇ are defined by (27).

Proof. Following the proof of Theorem 9, since $(0,0)$ is an extremal of $\bar{L}$ subject to the constraint $\bar{K} = 0$, the extended Lagrange multiplier rule (see for instance Theorem 4.1.3 of [van Brunt (2004)]) asserts the existence of reals λ_0 and λ, not both zero, such that $\nabla(\lambda_0 \bar{L}(0,0) - \lambda \bar{K}(0,0)) = \mathbf{0}$. Therefore,

$$\int_a^b \eta_1^\nabla(t)\left\{\lambda_0\left(\gamma_1 m^\rho(t) + \gamma_2 n(t)\right) - \lambda\left(k_1 f^\rho(t) + k_2 g(t)\right)\right\}\nabla t = 0. \tag{30}$$

Since (30) holds for any η_1, by Lemma 1 we have

$$\lambda_0\left(\gamma_1 m^\rho(t) + \gamma_2 n(t)\right) - \lambda\left(k_1 f^\rho(t) + k_2 g(t)\right) = c$$

for some $c \in \mathbb{R}$ and all $t \in [a,b]_\kappa$. This establishes equation (28). Equation (29) can be shown using a similar technique. □

Remark 13. *If $\hat{y} \in C^1_\diamond(I, \mathbb{R})$ is a normal extremizer to the isoperimetric problem* (8)–(10)*, then we can choose $\lambda_0 = 1$ in Theorem 12 and obtain Theorem 9. For abnormal extremizers, Theorem 12 holds with $\lambda_0 = 0$. The condition $(\lambda_0, \lambda) \neq \mathbf{0}$ guarantees that Theorem 12 is a useful necessary optimality condition.*

Example 14. *(abnormal extremal) Let $\mathbb{T} = \{1, 3, 4\}$ and consider the problem*

$$\text{minimize} \quad \mathcal{L}(y) = \int_1^4 t\left(y^\Delta(t)\right)^2 \Delta t \tag{31}$$

$$y(1) = 0, \quad y(4) = 1 \tag{32}$$

subject to the constraint

$$\mathcal{K}(y) = \int_1^4 t\left(y^\nabla(t)\right)^2 \nabla t = \frac{12}{11}. \tag{33}$$

Applying equation (28) *of Theorem 12 we get the following delta-nabla differential equation:*

$$\lambda_0 2\rho(t) y^{\Delta}(\rho(t)) - \lambda 2t y^{\nabla}(t) = C, \quad t \in \{3,4\},$$

where $C \in \mathbb{R}$*. By* (4) *we can write the above equation in the form*

$$\lambda_0 2\rho(t) y^{\nabla}(t) - \lambda 2t y^{\nabla}(t) = C, \quad t \in \{3,4\}. \tag{34}$$

Substituting $t = 3$ *and* $t = 4$ *into* (34) *we obtain*

$$\begin{cases} \lambda_0 y(3) - 3\lambda y(3) = C \\ 6\lambda_0(1 - y(3)) - 8\lambda(1 - y(3)) = C. \end{cases}$$

If we put $\lambda_0 = 1$*, then the above system of equations has no solutions. Therefore, we fix* $\lambda_0 = 0$*. In this case we obtain*

$$y_0(t) = \begin{cases} 0 & \text{if } t = 1 \\ \frac{8}{11} & \text{if } t = 3 \\ 1 & \text{if } t = 4 \end{cases}$$

as a candidate local minimizer to problem (31)–(33)*. Observe that* y_0 *is an extremal for* $\mathcal{K}$*. Indeed,*

$$\partial_3 K_{\Delta}[y_0](\rho(t)) - \int_a^{\rho(t)} \partial_2 K_{\Delta}[y_0](\tau)\Delta\tau \\ + \partial_3 K_{\nabla}\{y_0\}(t) - \int_a^t \partial_2 K_{\nabla}\{y_0\}(\tau)\nabla\tau = 2t y_0^{\nabla}(t) = \frac{24}{11}, \quad t \in \{3,4\}.$$

As a particular case of Theorem 12 we obtain the main result of [Almeida and Torres (2009)]:

Corollary 15 (Necessary optimality condition for normal and abnormal extremizers of a nabla isoperimetric problem – cf. Theorem 2 of Almeida and Torres (2009))**.** *Let* $\mathbb{T}$ *be a time scale,* $a, b \in \mathbb{T}$ *with* $a < b$*. If* $\hat{y}$ *is a local minimizer or maximizer to problem*

$$\text{extremize} \quad \int_a^b f(t, y^{\rho}(t), y^{\nabla}(t))\nabla t$$
$$\int_a^b g(t, y^{\rho}(t), y^{\nabla}(t))\nabla t = \Lambda$$
$$y(a) = \alpha\,, \quad y(b) = \beta$$

in the class of functions $y : [a,b] \to \mathbb{R}$ *such that* y^{∇} *exists and is continuous on* $[a,b]_{\kappa}$*, then there exist two constants* λ_0 *and* λ*, not both zero, such that*

$$\frac{\nabla}{\nabla t}\left[\partial_3 G\left(t, \hat{y}^{\rho}(t), \hat{y}^{\nabla}(t)\right)\right] - \partial_2 G\left(t, \hat{y}^{\rho}(t), \hat{y}^{\nabla}(t)\right) = 0$$

for all $t \in [a,b]_{\kappa}$*, where* $G(t,x,v) = \lambda_0 f(t,x,v) - \lambda g(t,x,v)$*.*

Proof. The result follows from Theorem 12 by considering the particular case $\gamma_1 = k_1 = 0$ and $\gamma_2 = k_2 = 1$. □

Other interesting corollaries are easily obtained from Theorem 12:

Corollary 16. (The delta-nabla Euler-Lagrange equations on time scales [Girejko et al. (2010)]). *If $\hat{y} \in C^1_\diamond(I, \mathbb{R})$ is a local extremizer to problem*

$$\begin{gathered} \textit{extremize} \quad \mathcal{L}(y) = \gamma_1 \int_a^b L_\Delta[y](t)\Delta t + \gamma_2 \int_a^b L_\nabla\{y\}(t)\nabla t \\ y(a) = \alpha\,, \quad y(b) = \beta \\ y \in C^1_\diamond(I, \mathbb{R})\,, \end{gathered}$$

then $\hat{y}$ satisfies the following delta-nabla integral equations:

$$\gamma_1 \left(\partial_3 L_\Delta[\hat{y}](\rho(t)) - \int_a^{\rho(t)} \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right) + \gamma_2 \left(\partial_3 L_\nabla\{\hat{y}\}(t) - \int_a^t \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const \tag{35}$$

for all $t \in I_\kappa$; and

$$\gamma_1 \left(\partial_3 L_\Delta[\hat{y}](t) - \int_a^t \partial_2 L_\Delta[\hat{y}](\tau)\Delta\tau\right) + \gamma_2 \left(\partial_3 L_\nabla\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} \partial_2 L_\nabla\{\hat{y}\}(\tau)\nabla\tau\right) = const$$

for all $t \in I^\kappa$.

Proof. The result follows from Theorem 12 by considering the particular case $k_1 = k_2 = k = 0$, for which the isoperimetric constraint (10) is trivially satisfied. □

3.2 DELTA-NABLA BI-OBJECTIVE PROBLEMS

We are now interested in studying the following bi-objective problem:

$$\text{minimize} \quad F(y) = \begin{bmatrix} \mathcal{L}_\Delta(y) \\ \mathcal{L}_\nabla(y) \end{bmatrix} \tag{36}$$

with

$$\begin{aligned} \mathcal{L}_\Delta(y) &= \int_a^b L_\Delta(t, y^\sigma(t), y^\Delta(t))\Delta t = \int_a^b L_\Delta[y](t)\Delta t\,, \\ \mathcal{L}_\nabla(y) &= \int_a^b L_\nabla(t, y^\rho(t), y^\nabla(t))\nabla t = \int_a^b L_\nabla\{y\}(t)\nabla t\,, \end{aligned}$$

and $y \in C^1_\diamond(I, \mathbb{R})$, $y(a) = \alpha$, $y(b) = \beta$, $t \in I$. A solution to this vector optimization problem is understood in the Pareto sense.

Definition 17 (locally Pareto optimal solution)**.** *A function* $\hat{y} \in C^1_\diamond(I,\mathbb{R})$ *is called* a local Pareto optimal solution *if there exists* $\delta > 0$ *for which does not exist* $y \in C^1_\diamond(I,\mathbb{R})$ *with* $||\hat{y} - y||_{1,\infty} < \delta$ *and*

$$\mathcal{L}_\Delta(y) \leq \mathcal{L}_\Delta(\hat{y}) \quad \wedge \quad \mathcal{L}_\nabla(y) \leq \mathcal{L}_\nabla(\hat{y}),$$

where at least one of the above inequalities is strict.

Theorem 18 (Necessity)**.** *If* $\hat{y}$ *is a local Pareto optimal solution to the bi-objective problem* (36)*, then* $\hat{y}$ *is a minimizer to the isoperimetric problems*

$$\textit{minimize} \;\; \mathcal{L}_\Delta(y) \;\; \textit{subject to} \;\; \mathcal{L}_\nabla(y) = \mathcal{L}_\nabla(\hat{y})$$

and

$$\textit{minimize} \;\; \mathcal{L}_\nabla(y) \;\; \textit{subject to} \;\; \mathcal{L}_\Delta(y) = \mathcal{L}_\Delta(\hat{y})$$

simultaneously.

Proof. A proof can be done similarly to the proof of Theorem 3.8 in [Malinowska and Torres (2009b)]. □

Example 19. *Let us consider* $\mathbb{T} = \{1,3,4\}$ *and the bi-objective optimization problem* (36) *with*

$$\begin{aligned} \mathcal{L}_\Delta(y) &= \int_1^4 t\left(y^\Delta(t)\right)^2 \Delta t, \\ \mathcal{L}_\nabla(y) &= \int_1^4 t\left(y^\nabla(t)\right)^2 \nabla t. \end{aligned} \tag{37}$$

We pose the question of finding local Pareto optimal solutions to (37) *under the boundary conditions*

$$y(1) = 0, \quad y(4) = 1. \tag{38}$$

Let us consider the following function

$$\hat{y}(t) = \begin{cases} 0 & \textit{if } t = 1 \\ \frac{9}{11} & \textit{if } t = 3 \\ 1 & \textit{if } t = 4. \end{cases}$$

As it is shown in Example 10, $\hat{y}$ *is, simultaneously, a candidate minimizer to the problem*

$$\begin{gathered} \textit{minimize} \;\; \mathcal{L}_\nabla(y) = \int_1^4 t\left(y^\nabla(t)\right)^2 \nabla t \\ y(1) = 0, \quad y(4) = 1 \end{gathered}$$

subject to

$$\mathcal{L}_\Delta(y) = \int_1^4 t\left(y^\Delta(t)\right)^2 \Delta t = \frac{105}{242}.$$

and

$$\text{minimize} \quad \mathcal{L}_\Delta(y) = \int_1^4 t\left(y^\Delta(t)\right)^2 \Delta t$$
$$y(1) = 0, \quad y(4) = 1$$

subject to

$$\mathcal{L}_\nabla(y) = \int_1^4 t\left(y^\nabla(t)\right)^2 \nabla t = \frac{25}{22}.$$

According to Theorem 18, the function $\hat{y}$ is a candidate Pareto optimal solution to the bi-objective problem (37)–(38)*.*

Theorem 18 shows that necessary optimality conditions to isoperimetric problems (see Section 3.1) are also necessary to local Pareto optimality of a bi-objective variational problem on time scales. Indeed, functional (8) in particular cases when $\gamma_1 = 1$ and $\gamma_2 = 0$ or $\gamma_1 = 0$ and $\gamma_2 = 1$ is reduced either to $\mathcal{L}(y) = \mathcal{L}_\Delta(y)$ or to $\mathcal{L}(y) = \mathcal{L}_\nabla(y)$.

The next theorem asserts that sufficient conditions of optimality for scalar optimal control problems are also sufficient conditions for Pareto optimality.

Theorem 20 (Sufficiency)**.** *A local minimizer $\hat{y}$ to the functional $\gamma\mathcal{L}_\Delta(y) + (1-\gamma)\mathcal{L}_\nabla(y)$ with $\gamma \in (0,1)$ is a local Pareto optimal solution to the bi-objective problem* (36)*.*

Proof. A proof can be done similarly to the proof of Theorem 3.7 in [Malinowska and Torres (2009b)]. □

Example 21. a) *Let us consider $\mathbb{T} = \{1,3,4\}$ and the bi-objective optimization problem* (36) *defined by*

$$\begin{aligned} \mathcal{L}_\Delta(y) &= \textstyle\int_1^4 t\left(y^\Delta(t)\right)^2 \Delta t, \\ \mathcal{L}_\nabla(y) &= \textstyle\int_1^4 t\left(y^\nabla(t)\right)^2 \nabla t \end{aligned} \tag{39}$$

subject to

$$y(1) = 0, \quad y(4) = 1. \tag{40}$$

By Theorem 20 we can find Pareto optimal solutions to this problem by considering the family of problems

$$\min \quad \gamma\mathcal{L}_\Delta(y) + (1-\gamma)\mathcal{L}_\nabla(y)$$
$$y(1) = 0, \quad y(4) = 1,$$

where $\gamma \in (0,1)$. Using condition (35) *of Corollary 16 we get the following equation:*

$$2\gamma\rho(t)y^\Delta(\rho(t)) + 2(1-\gamma)ty^\nabla(t) = c \quad \forall t \in \{3,4\} \tag{41}$$

for some $c \in \mathbb{R}$*. Substituting* $t = 3$ *and* $t = 4$ *into* (41)*, we obtain*

$$\gamma y(3) + 3(1-\gamma)y(3) = c,$$
$$6\gamma(1 - y(3)) + 8(1-\gamma)(1 - y(3)) = c\,,$$

and from this we have $y(3) = \frac{8-2\gamma}{11-4\gamma}$*,* $\gamma \in (0,1)$*. Since* $L_\Delta(\cdot,\cdot,\cdot)$ *and* $L_\nabla(\cdot,\cdot,\cdot)$ *are jointly convex with respect to the second and third argument for any* $t \in \mathbb{T}$*, the local Pareto optimal solutions to problem* (39)–(40) *are*

$$y(t) = \begin{cases} 0, & \text{if } t = 1 \\ k, & \text{if } t = 3, \quad k \in \left(\frac{8}{11}, \frac{6}{7}\right). \\ 1, & \text{if } t = 4, \end{cases}$$

b) *Let us consider* $\mathbb{T} = \{0, 1, 2\}$ *and the bi-objective problem* (36) *defined by*

$$\begin{aligned} \mathcal{L}_\Delta(y) &= \int_0^2 \left(y^\sigma(t)\right)^2 \Delta t, \\ \mathcal{L}_\nabla(y) &= \int_0^2 \left(y^\rho(t) - 3\right)^2 \nabla t \end{aligned} \tag{42}$$

subject to

$$y(0) = 0, \quad y(2) = 0. \tag{43}$$

By Theorem 20 we can find Pareto optimal solutions to this problem by considering the family of problems

$$\begin{gathered} min \quad \gamma\mathcal{L}_\Delta(y) + (1-\gamma)\mathcal{L}_\nabla(y) \\ y(0) = 0, \quad y(2) = 0, \end{gathered}$$

where $\gamma \in (0,1)$*. Using condition* (35) *of Corollary 16 we get the following equation:*

$$\gamma \int_0^{\rho(t)} y^\sigma(\tau)\Delta\tau + (1-\gamma)\int_0^t (y^\rho(\tau) - 3)\nabla\tau = c \quad \forall t \in \{1, 2\} \tag{44}$$

for some $c \in \mathbb{R}$*. Substituting* $t = 1$ *and* $t = 2$ *into* (44) *we obtain*

$$\gamma \int_0^0 y^\sigma(\tau)\Delta\tau + (1-\gamma)\int_0^1 (y^\rho(\tau) - 3)\nabla\tau = c,$$
$$\gamma \int_0^1 y^\sigma(\tau)\Delta\tau + (1-\gamma)\int_0^2 (y^\rho(\tau) - 3)\nabla\tau = c\,,$$

and from this we have $y(1) = 3 - 3\gamma$*,* $\gamma \in (0,1)$*. Since* $L_\Delta(\cdot,\cdot,\cdot)$ *and* $L_\nabla(\cdot,\cdot,\cdot)$ *are jointly convex with respect to the second and third argument for any* $t \in \mathbb{T}$*, the local Pareto optimal solutions to problem* (42)–(43) *are*

$$y(t) = \begin{cases} 0, & \text{if } t = 0 \\ k, & \text{if } t = 1, \quad k \in (0, 3). \\ 0, & \text{if } t = 2, \end{cases}$$

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